Altruism in groups

Ilaria Brunetti, Rachid El-Azouzi, Eitan Altman

Chapter 2

Altruism in groups

Ilaria Brunetti, Rachid El-Azouzi and Eitan Altman

Evolutionary Game Theory has been originally developed and formalized by [247], in order to model the evolution of animal species and it has soon become an important mathematical tool to predict and even design evolution in many fields, others than biology. It mainly focuses on the dynamical evolution of the strategies adopted in a population of interacting individuals, where the notion of equilibrium adopted is that of Evolutionarily Stable Strategy (ESS, [247]), implying robustness against a mutation (i.e. a change in the strategy) of a small fraction of the population. This is a stronger condition than the standard Nash equilibrium concept, which requires robustness against deviation of a single user. On the importance of the ESS for understanding the evolution of species, Dawkins writes in his book "The Selfish Gene" [290]: "we may come to look back on the invention of the ESS concept as one of the most important advances in evolutionary theory since Darwin." He further specifies: "Maynard Smith's concept of the ESS will enable us, for the first time, to see clearly how a collection of independent selfish entities can come to resemble a single organized whole.

Evolutionary game theory is nowadays considered as an important enrichment of game theory and it's applied in a wide variety of fields, spanning from social sciences [107] to computer science. Some examples of applications in computer science can be found in multiple access protocols [250], multihoming[240] and resources competition in the Internet [298].

This theory is usually adopted in situations where individuals belonging to a very large population are matched in random pairwise interactions. In classical evolutionary games (EG), each individual constitutes a selfish player involved in a non-cooperative game, maximizing its own utility, also said fitness, since in

EG it's originally assumed that individuals' utility corresponds to the Darwinian fitness, i.e. the number of offsprings. The fitness is defined as a function of both the behavior (strategy) of the individual as well as of the distribution of behaviors among the whole population, and strategies with higher fitness are supposed to spread within the population. A behavior of an individual with a higher fitness would thus result in a higher rate of its reproduction. We observe that since strategies and fitness are associated to the individual, then classical EG is restricted to describe populations in which the individual is the one that is responsible for the reproduction and where the choice of its own strategies is completely selfish. In biology, in some species like bees or ants, the one who interacts is not the one who reproduces. This implies that the Darwinian fitness is related to the entire swarm and not to a single bee and thus, standard EG models excludes these species in which the single individual which reproduces is not necessarily the one that interacts with other players. Furthermore, in many species, we find altruistic behaviors, which favors the group the playing individual belongs to, but which may hurt the single individual. Altruistic behaviors are typical of parents toward their children: they may incubate them, feed them or protect them from predator's at a high cost for themselves. Another example can be found in flock of birds: if a bird sees a predator, then it gives an alarm call to warn the rest of the flock to protect the group, but attracting the predators attention to itself. Also the stinging behavior of bees is another example of altruism, since it serves to protect the hive but its lethal for the bee which strives. In human behavior, many phenomena where individuals do care about other's benefits in their groups or about their intentions can be observed and thus the assumption of selfishness becomes inconsistent with the many real world behavior of individuals belonging to a population.

Founders of classical EG seem to have been well aware of this problem. Indeed, Vincent writes in [278] "Ants seem to completely subordinate any individual objectives for the good of the group. On the other hand, the social foraging of hyenas demonstrates individual agendas within a tight-knit social group (Hofer and East, 2003). As evolutionary games, one would ascribe strategies and payoffs to the ant colony, while ascribing strategies and payoffs to the individual hyenas of a pack."

In the case of ants, the proposed solution is thus to model the ant colony as a player. Within the CEG paradigm, this would mean that we have to consider interactions between ant colonies. The problem of this approach is that it doesn't allow to model behavior at the level of the individual.

In this chapter we present a new model for evolutionary games in which the concept of the agent as a single individual is replaced by that of the agent as a whole group of individuals. We define a new equilibrium concept, named Group Equilibrium Stable Strategy (GESS), allowing to model competition between individuals in a population in which the whole group shares a common utility. Even if we still consider pairwise interactions among individuals, our perspective is substantially different: we assume that individuals are simple actors of the game, maximizing the utility of their group instead of their own one. We first define the GESS, deriving it in several ways and exploring its major characteristics. The main interest of this work is to study how this new concept changes the profile of population and to explore the relationship between GESS and standard Nash equilibrium and ESS. We characterize the GESS and we show how the evolution and the equilibrium are influenced by the groups' size as well as by their immediate payoff. We compute some interesting in a particular example of a multiple access games, in which the payoff related to local interactions also depend on the type of individuals that are competing, and not only the strategy used. In such application, we evaluate the impact of altruism behavior on the performance of the system.

The chapter is structured as follows. We first provide in section 2.1 the needed background on evolutionary games. In the section 2.2 we then study the new natural concept GESS and the relationship between GESS and ESS or Nash equilibrium. The characterization of the GESS is studied in section 2.3. Section 2.4 provides some numerical illustration through some famous examples in evolutionary games. In section 2.5 we study the multiple access control in slotted Aloha under altruism behavior. The paper closes with a summary in section 5.6

2.1 Classical Evolutionary Games and ESS

We consider an infinite population of players and we assume that each individual disposes of the same set of pure strategies $\mathcal{K} = \{1, 2, .., m\}$. Each individual repeatedly interacts with an other individual randomly selected within the population. Individuals may use a mixed strategy $\mathbf{p} \in \Delta(\mathcal{K})$, where $\Delta(\mathcal{K}) = \{\mathbf{p} \in \mathbb{R}_+^m | \sum_{i\in\mathcal{K}} p_i = 1\}$, where each vector $\mathbf{p}$ corresponds to a probability measure over the set of actions $\mathcal{K}$. This serves to represent those cases where an individual has the capacity to produce a variety in behaviors. A mixed strategy $\mathbf{p}$ can also be interpreted as the vector of densities of individuals adopting a certain pure strategy, such that each component p_i represents the fraction of the population using strategy $i \in \mathcal{K}$. However, in the original formulation of evolutionary game theory, it is not necessary to make the distinction between population-level and individual-level variability for infinite population [247].

Let now focus on the case of monomorphic populations, i.e., on the case in which individuals use mixed strategy. We define by $J(\mathbf{p}, \mathbf{q})$ the expected payoff of a given individual if it uses a mixed action $\mathbf{p}$ when interacting with an individual playing the mixed action $\mathbf{q}$. Actions with higher payoff (or "fitness") are thus expected to spread faster in the population. If we define a payoff matrix A and consider $\mathbf{p}$ and $\mathbf{q}$ to be column vectors, then $J(\mathbf{p}, \mathbf{q}) = \mathbf{p}' A\mathbf{q}$ and the payoff function J is bilinear, i.e. it is linear both in $\mathbf{p}$ and in $\mathbf{q}$. A mixed action $\mathbf{q}$ is said to be a Nash equilibrium if

$$\forall \mathbf{p} \in \Delta(\mathcal{K}), \quad J(\mathbf{q}, \mathbf{q}) \geq J(\mathbf{p}, \mathbf{q}) \tag{2.1}$$

As we mentioned above, in evolutionary games the most important concept of equilibrium is the ESS, introduced by [247] as a strategy that, if adopted by the whole population, can not be invaded by a different ("mutant") strategy. More precisely, if we suppose that the entire population adopt a strategy $\mathbf{q}$ and that a small fraction ϵ of individuals (*mutants*) plays another strategy $\mathbf{p}$, then $\mathbf{q}$ is evolutionarily stable against $\mathbf{p}$ if

$$J(\mathbf{q}, \epsilon\mathbf{p} + (1-\epsilon)\mathbf{q}) > J(\mathbf{p}, \epsilon\mathbf{p} + (1-\epsilon)\mathbf{q}) \tag{2.2}$$

The definition of ESS thus corresponds to a robustness property against deviations by a (small) fraction of the population. This is an important difference that distinguishes the equilibrium in a large population as seen in evolutionary games and the standard Nash equilibrium often used in economic context, where the robustness is defined against the possible deviation of each single agent. Since in evolutionary games context we deal with very large populations, it is more likely to expect that some group of individuals may deviate from the incumbent strategy and thus robustness against deviations by a single user would not be sufficient to guarantee that the mutant strategy will not spread among a growing portion of the population. By defining the ESS through the following equivalent definition, it's possible to establish the relationship between ESS and Nash Equilibrium (NE). The proof of the equivalence between the two definitions can be found in [280, Proposition 2.1] or [143, Theorem 6.4.1, page 63].
Strategy $\mathbf{q}$ is an ESS if it satisfies the two conditions:

- Nash equilibrium condition:

$$J(\mathbf{q}, \mathbf{q}) \geq J(\mathbf{p}, \mathbf{q}) \qquad \forall \mathbf{p} \in \mathcal{K}. \tag{2.3}$$

- Stability condition:

$$J(\mathbf{p}, \mathbf{q}) = J(\mathbf{q}, \mathbf{q}) \Rightarrow J(\mathbf{p}, \mathbf{p}) < J(\mathbf{q}, \mathbf{p}). \quad \forall \mathbf{p} \neq \mathbf{q} \tag{2.4}$$

The first condition (2.3) corresponds to the condition for a Nash equilibrium. In fact, if inequality (2.3) is satisfied, then the fraction of mutations in will increase, since it hasn't an higher fitness, and thus a lower growth rate. If the two strategies provide the same fitness but condition (2.4) holds, then a population using q is "weakly" immune against mutants using p. Indeed, if the mutant's population grows, then we shall frequently have individuals with action q competing with mutants. In such cases, the inequality in (2.4), $J(\mathbf{p}, \mathbf{p}) < J(\mathbf{q}, \mathbf{p})$, ensures that the growth rate of the original population exceeds that of the mutants. In this sense an ESS can be seen as a refinement of the Nash equilibrium.

2.2 New natural concept on evolutionary games

In this section we present our new concept for evolutionary games, where we still consider pairwise interactions among individuals but the actual player of the game is a whole group of these individuals. We suppose that the population is composed of N groups, G_i, $i = 1, 2, .., N$, where the normalized size of each group G_i is denoted by α_i, with $\sum_{j=1}^{N} \alpha_i = 1$.

Each individual can meet a member of its own group or of a different one, in random pairwise interactions, and disposes of a finite set of actions, denoted by $\mathcal{K} = \{a_1, a_2, .., a_M\}$. Let p_{ik} be the probability that an individual in the group G_i chooses an action $a_k \in \mathcal{K}$. Each group i is associated to the vector of probabilities $\mathbf{p_i} = (p_{i1}, p_{i2}, .., p_{iM})$ where $\sum_{l=1}^{M} p_{il} = 1$, giving the distribution of actions within the group. By assuming that each individual can interact with any other individual with equal probability, then the expected utility of a player (the group) i is:

$$U_i(\mathbf{p}_i, \mathbf{p}_{-i}) = \sum_{j=1}^{N} \alpha_j J(\mathbf{p}_i, \mathbf{p}_j), \tag{2.5}$$

where $\mathbf{p}_{-i}$ is the strategies profile of all the other groups (but i) and $J(\mathbf{p}_i, \mathbf{p}_j)$ is the immediate expected utility of an individual player adopting strategy $\mathbf{p}_i$ against an opponent playing $\mathbf{p}_j$.

2.2.1 Group Equilibrium Stable Strategy

The definition of GESS is related to a notion of robustness against deviations within each group. In this context, there are two possible equivalent interpretations of an $\epsilon-$ deviation toward $\mathbf{p}_i$. If the group G_i plays according to the incumbent strategy $\mathbf{q}_i$, an $\epsilon-$ deviation, can be thought as:

1. A small deviation in the strategy by all members of a group, shifting to the new group's strategy $\bar{\mathbf{p}}_i = \epsilon \mathbf{p}_i + (1 - \epsilon \mathbf{q}_i)$;

2. The second is a (possibly large) deviation of a fraction ϵ of individuals belonging to G_i, playing the mutant strategy $\mathbf{p}_i$.

After an $\epsilon-$deviation under both interpretations the profile of the whole population becomes $\alpha_i \epsilon \mathbf{p}_i + \alpha_i(1-\epsilon)\mathbf{q}_i + \sum_{j \neq i} \alpha_j \mathbf{q}_j$. Then the average payoff of group G_i after mutation is given by:

$$\begin{aligned} U_i(\bar{\mathbf{p}}_i, \mathbf{q}_{-i}) &= \sum_{j=1}^{N} \alpha_j J(\bar{\mathbf{p}}_i, \mathbf{p}_j) \\ &= U_i(\mathbf{q}_i, \mathbf{q}_{-i}) + \epsilon^2 \alpha_i \Omega(\mathbf{p}_i, \mathbf{q}_i) + \epsilon\Big(\alpha_i (J(\mathbf{p}_i, \mathbf{q}_i) \\ &+ J(\mathbf{q}_i, \mathbf{p}_i) - 2J(\mathbf{q}_i, \mathbf{q}_i)) + \sum_{j \neq i} (J(\mathbf{p}_i, \mathbf{q}_j) - J(\mathbf{q}_i, \mathbf{q}_j)\Big) \end{aligned} \tag{2.6}$$

where $\Omega(\mathbf{p}_i, \mathbf{q}_i) := J(\mathbf{p}_i, \mathbf{p}_i) - J(\mathbf{p}_i, \mathbf{q}_i) - J(\mathbf{q}_i, \mathbf{p}_i)) + J(\mathbf{q}_i, \mathbf{q}_i)$.

Definition 1. *A strategy* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, .., \mathbf{q}_N)$ *is a GESS if* $\forall i \in \{1, \ldots, N\}$, $\forall \mathbf{p}_i \neq \mathbf{q}_i$, *there exists some* $\epsilon_{\mathbf{p}_i} \in (0, 1)$, *which may depend on* $\mathbf{p}_i$, *such that for all* $\epsilon \in (0, \epsilon_{\mathbf{p}_i})$

$$U_i(\bar{\mathbf{p}}_i, \mathbf{q}_{-i}) < U_i(\mathbf{q}_i, \mathbf{q}_{-i}), \tag{2.7}$$

where $\bar{\mathbf{p}}_i = \epsilon \mathbf{p}_i + (1 - \epsilon)\mathbf{q}_i$.

From equation (2.7), this implies that strategy $\mathbf{q}$ is a GESS if the two following conditions hold:

- $\forall \mathbf{p}_i \in [0,1]^M$

$$F_i(\mathbf{p}_i, \mathbf{q}) := \alpha_i \Omega(\mathbf{p}_i, \mathbf{q}_i) - U_i(\mathbf{p}_i, \mathbf{q}_{-i}) + U_i(\mathbf{q}_i, \mathbf{q}_{-i}) \geq 0, \tag{2.8}$$

- $\exists \mathbf{p}_i \neq \mathbf{q}_i$ such that:

$$\text{If} \quad F_i(\mathbf{p}_i, \mathbf{q}) = 0 \Rightarrow \Omega(\mathbf{p}_i, \mathbf{q}_i) < 0 \tag{2.9}$$

Remark 1. *The second condition (2.9) can be rewritten as*

$$U_i(\mathbf{q}_i, \mathbf{q}_{-i}) > U_i(\mathbf{p}_i, \mathbf{q}_{-i})$$

which coincide to the definition of the strict Nash equilibrium of the game composed by N groups, each of them maximizing its own utility.

2.2.2 GESS and standard ESS

Here we analyze the relationship between our new equilibrium concept, the $GESS$ and the standard ESS.

Proposition 1. *Consider games such that the immediate expected reward is symmetric, i.e. $J(\mathbf{p}, \mathbf{q}) = J(\mathbf{q}, \mathbf{p})$. Then any ESS is a GESS.*

Proof. Let $\mathbf{q} = (q, .., q)$ be an ESS. From the symmetry of the payoff function and equation (2.8), we get:

$$\begin{aligned}
F_i(\mathbf{p}_i, \mathbf{q}) &= -\Big(\alpha_i (J(\mathbf{p}_i, q) + J(q, \mathbf{p}_i) - 2J(q,q) \\
&+ \sum_{j \neq i} \alpha_j (J(\mathbf{p}_i, q) - J(q,q)\Big) \\
&= -2\alpha_i (J(\mathbf{p}_i, q) - J(q,q)) - \sum_{j \neq i} \alpha_j (J(\mathbf{p}_i, q) - J(q,q)) \\
&= -(1 + \alpha_i)(J(\mathbf{p}_i, q) - J(q,q)) \geq 0
\end{aligned}$$

where the second equality follows from the symmetry of J and the last inequality follows form the fact that $\mathbf{q}$ is an ESS and satisfies (2.3). This means that $\mathbf{q}$ satisfies the first condition of GESS (2.8). Assume now that $F_i(\mathbf{p}_i, q) = 0$ for some $\mathbf{p}_i \neq \mathbf{q}$, previous equations imply that $J(\mathbf{p}_i, \mathbf{q}) = J(\mathbf{q}, \mathbf{q})$. Thus the second condition (2.9) becomes $\Omega(\mathbf{p}_i, \mathbf{q}_i) = J(\mathbf{p}_i, \mathbf{q}) - J(\mathbf{q}, \mathbf{q}) < 0$ which coincide with the second condition of ESS (2.4). This completes the proof. □

2.2.3 Nash equilibrium and GESS

In the classical evolutionary games, the ESS can be seen as a refinement of a Nash equilibrium, since all ESSs are Nash equilibria while the converse is not true. In order to study and characterize this relationship in our context, we define the game between groups. There are N players in which each player has a finite set of pure strategies $\mathcal{K} = \{1, 2, .., m\}$; let $U_i(\mathbf{q}_i, \mathbf{q}_{-i})$ be the utility of player i when using mixed strategy $\mathbf{q}_i$ against a population of players using $\mathbf{q}_{-i} = (\mathbf{q}_1, \dots, \mathbf{q}_{i-1}, \mathbf{q}_{i+1}, \dots, \mathbf{q}_N)$.

Definition 2. *A strategy $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, .., \mathbf{q}_N)$ is a Nash Equilibrium if $\forall i \in \{1, \dots, N\}$*

$$U_i(\mathbf{q}_i, \mathbf{q}_{-i}) \geq U_i(\mathbf{p}_i, \mathbf{q}_{-i}) \tag{2.10}$$

for every mixed strategy $\mathbf{p}_i \neq \mathbf{q}_i$. If the inequality is a stric one, then $\mathbf{q}$ is a strict Nash equilibrium.

From the definition of the strict Nash equilibrium, it is easy to see that any strict Nash equilibrium is a GESS defined in equation (2.7). But in our context, we address several questions on the relationship between the GESS, ESS and the Nash equilibrium defined in (2.10). For the sake of simplicity, we restrict to the case of two-strategies games. Before studying them, we introduce here some necessary definitions.

Definition 3.

- *A fully mixed strategy $\mathbf{q}$ is a strategy such that all the actions available for a group have positive probability, i.e., $0 < q_{ij} < 1$ $\forall (i, j) \in \mathcal{I} \times \mathcal{K}$.*
- *A mixer (resp. pure) group i is the group that uses a mixed (resp. pure) strategy $0 < q_i < 1$ (resp. $q_i \in \{0, 1\}$).*
- *An equilibrium with mixed and non mixed strategies is an equilibrium in which there is at least one pure group and a mixer group.*

2.3 Analysis of N-groups games with two strategies

In this section we present a simple case, with N-groups games disposing of two strategies. The two available pure strategies are A and B and the payoff matrix of pairwise interctions is given by:

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array},$$

where $P_{ij},\ i, j = A,\ B$ is the payoff of the first (row) individual if it plays strategy i against the second (column) individual playing strategy j. We assume that the two individuals are symmetric and hence payoffs of the column player are given by P^t, i.e. the transposed of P. According to the definition of GESS, $\mathbf{q}$ is a GESS if it satisfies the conditions (2.8)-(2.9), which can be rewritten here as:

- $\forall \mathbf{p}_i \in [0,1]$, $i = 1, .., N$:

$$F_i(p_i, \mathbf{q}) = (q_i - p_i)\Big(\alpha_i(J(q_i,1) - J(q_i,0)) + \sum_{j=1}^{N} \alpha_j(J(1,q_j) - J(0,q_j))\Big) \geq 0 \tag{2.11}$$

- If $F(p_i, \mathbf{q}) = 0$ for some $p_i \neq q_i$, then:

$$(p_i - q_i)^2 \Delta < 0 \Longrightarrow \Delta < 0 \tag{2.12}$$

  where $\Delta = a - b - c + d$.

2.3.1 Characterization of fully mixed GESS

In this section we are interested in characterizing the full mixed GESS $\mathbf{q}$. According to (4.4), a full mixed equilibrium $\mathbf{q} = (q_1, \ldots, q_N)$ is a GESS if it satisfies the condition (4.5) where the equality must holds for all $p \in [0,1]$. This yields to the following equation: $\forall i = 1, \ldots N$,

$$\alpha_i(J(q_i,1) - J(q_i,0)) + \sum_{j=1}^{N} \alpha_j(J(1,q_j) - J(0,q_j)) = 0$$

which can be rewritten as

$$\alpha_i \Delta q_i + b - d + \alpha_i(c-d) + \Delta \sum_{j=1}^{N} \alpha_j q_j = 0$$

This leads to the following expression of the mixed GESS:

$$q_i^* = \frac{d - b + \big((1+N)\alpha_i - 1\big)(d-c)}{(N+1)\alpha_i \Delta}; \tag{2.13}$$

Proposition 2. *If $\Delta < 0$ and $0 < q_i^* < 1$, $i = 1, \ldots, N$, then there exists a unique fully mixed GESS equilibrium given by (2.13).*

Note that the fully mixed GESS is a strict Nash equilibrium since the condition (4.5) corresponds to the definition of the strict Nash equilibrium (see remark 1), under the condition $F(p, q_1, \ldots, q_N) = 0$, $\forall p \in [0,1]$.

2.3.2 Characterization of strong GESS

We say that an equilibrium is a strong GESS if it satisfies the strict inequality (4.4) for all groups. As we did above for the fully mixed GESS, we expicit here the condition for the existence of a strong GESS. Note that all groups have to use pure strategy in a strong GESS. With no loss of generality, we assume that a pure strong GESS can be represented by n_A, where $n_A \in \{1, ..., N\}$ denotes that the n_A first groups use A pure strategy and remaining $N - n_A$ groups chose strategy B. For example $n_A = N$ (resp. $n_A = 0$) means that all groups choose pure strategy A (resp. B).

Proposition 3. *If $a \neq c$ or $b \neq d$, then every N-player game with two strategies has a GESS. We distinguish the following possibilities for the* ***strong GESS****:*

i. If $a - c > \max_i(\alpha_i) \cdot (b - a)$ then $n_A = N$ is a strong GESS;

ii. If $b - d < \min_i(\alpha_i)(d - c)$ then $n_A = 0$ is a strong GESS;

iii. Let $H(n_a) := \sum_{j=1}^{n_A} \alpha_j(a-c)+\sum_{j=n_A+1}^{N} \alpha_j(b-d)$. If $\alpha_i(d-c) > H(n_a) > \alpha_i(b-a)$ then n_A is a strong GESS.

Proof. In order to prove that a strategy $n_A = N$ is a GESS, we have to impose the strict inequality, i.e.: $\forall p_i \neq 1$ for $i \in \{1, .., n_A\}$ and $\forall p_i \neq 0$ for $i \in \{n_A + 1, .., N\}$

$$F_i(p_i, 1_{n_A}, 0_{N-n_A}) > 0$$

We provide here the necessary conditions of the existence in the case $n_A = N$; the other cases straightforward follow from the players' symmetry assumption.

We now suppose that $(n_A = N)$ is a strong GESS. The inequality (4.4) becomes: $\forall p_i \neq 1, \ \forall i$

$$(p_i - 1)\Big(\alpha_i(a - b) + \sum_{j=1}^{N} \alpha_j(a - c)\Big) = (p_i - 1)\Big(\alpha_i(a - b) + a - c)\Big) < 0,$$

Since $p_i < 1$, one has $\alpha_i(b - a) < a - c \ \forall i$. This completes the proof of (i). To show conditions of the other strong GESSs, we follow the lines of the proof of (i). □

2.3.3 Characterization of weak GESS

We define a weak GESS as an equilibrium in which at least one group uses a strategy that satisfies the condition (4.5) with equality. Here we distinguish two different situations: the equilibrium with no mixer group and the equilibrium with mixer and no mixer groups. Conditions for the equilibrium with no mixed strategy are given by Proposition 3 with at the least one group satisfying it with equality and $\Delta < 0$. In what follows we thus focus only on the equilibrium with mixer and no mixer groups. Without loss of generality, we assume that an equilibrium with mixed and non mixed strategies, can be represented by $(n_A, n_B, \mathbf{q})$ where n_A denotes that group i for $i = 1.., n_A$ (resp. $i = n_A + 1, .., n_A + n_B$) uses strategy A (resp. B). The remaining groups $N - n_A - n_B$ are mixers in which q_i is the probability to choose the strategy A by group i.

Proposition 4. *Let either $a \neq c$ or $b \neq d$ and $\Delta < 0$. $(n_A, n_B, \mathbf{q})$ is a weak GESS if:*

$$\begin{cases} \alpha_i\Delta + d - b + \alpha_i(c - d) + \Delta(\alpha_{n_A} + y) \geq 0, \ i = 1, .., n_A \\ d - b + \alpha_i(c - d) + \Delta(\alpha_{n_A} + y) \leq 0, \ i = n_A + 1, .., n_B \\ q_i = \frac{d-b+\alpha_i(d-c)-y\Delta}{\Delta\alpha_i}, \ i = n_A + n_B + 1, .., N \end{cases} \tag{2.14}$$

where $y = \frac{(N-n_A-n_B)(d-b-\sum_{j=1}^{n_A} \alpha_j)+(d-c)\sum_{j=n_A+n_B+1}^{N} \alpha_j}{\Delta(N-N_A-n_B+1)}$.

Proof. See technical report □

2.4 Some Examples

In this section we analyze a number of examples with two players and two strategies.

2.4.1 Hawk and Dove Game

One of the most studied examples in EG theory is the Hawk-Dove game, first introduced by Maynard Smith and Price in "The Logic of Animal Conflict". This game serves to represent a competition between two animals for a resource. Each animal can follow one of two strategies, either Hawk or Dove, where Hawk corresponds to an aggressive behavior, Dove to a non-aggressive one. So, if two Hawks meet, they involve in a fight, where one of them obtain the resource and the other is injured, with equal probability. A Hawk always wins against a Dove, but there is no fight, so the Dove looses the resource but it's not injured, whereas if two Doves meet they equally share the resource. The payoff matrix associated to the game is the following:

$$\begin{array}{cc} & \begin{array}{cc} H & D \end{array} \\ \begin{array}{c} H \\ D \end{array} & \begin{pmatrix} \frac{1}{2}(V-C) & V \\ 0 & V/2 \end{pmatrix} \end{array}$$

where C represent the cost of the fight, and V is the benefit associated to the resource. We assume that $C > V$.

In standard GT, this example belongs to the class of *anti-coordination games*, which always have two strict pure strategy NEs and one non-strict, mixed strategy NE. In this case the two strict pure equilibria are (H, D) and (D, H), and the mixed-one is: $q^* = \frac{V}{C}$. The latter is the only ESS: even if the two pure NE are strict, being asymmetric they can't be ESSs. We now set $V = 2$ and $C = 3$ and we study the Hawk and Dove game in our groups framework, considering two groups of normalized size α and $1 - \alpha$.

We find that the GESSs and the strict NE always coincides. More precisely we obtain that:

- for $0 < \alpha < 0.25$ the game has one strong GESS (H, D) and a weak GESS (H, q_2) ;
- for $0.25 < \alpha < 0.37$: one weak GESS (H, q_2);
- for $0.37 < \alpha < 0.5$ one weak GESS, (q_1^*, q_2^*);

We observe that the size of groups has a strong impact on the behavior of players: in the first interval of $\alpha-$values we remark that the GESS is not unique; if the size of the first group increases (which implies that the second one decreases), the probability that the second group plays aggressively against the pure aggressive strategy of the first one increases until we get into the third interval, where both players are mixers. The mixed equilibrium q_1^* is decreasing in α: this is because, as we supposed that an individual can interact with members of its own group, when increasing α, the probability of meeting an individual in the same group increases and thus the individual tend to adopt a less aggressive behavior.

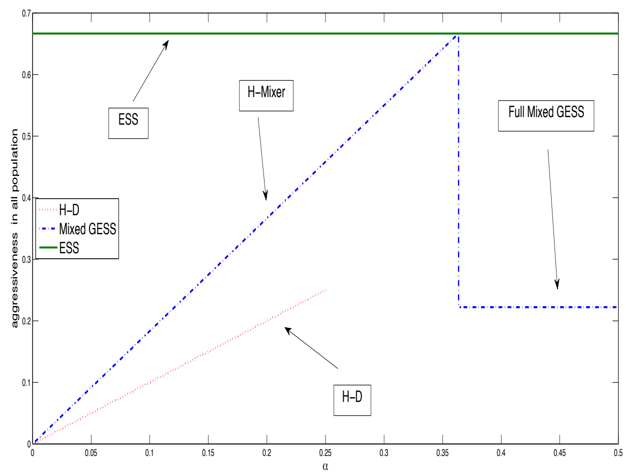

Figure 2.1: The global level of aggressiveness in the two-groups population for the different GESSs, as a function of α

2.4.2 Stag Hunt Game

We now consider a well-known example which belonging to the class of *coordination games*, the Stag Hunt game. The story modeled by this game has been described by J-J. Rousseau: two individuals go hunting; if they hunt together cooperating, they can hunt a stag; otherwise, if hunting alone, a hunter can only get a hare. In this case, collaboration is thus rewarding for players. It serves to represent a conflict between social and safely cooperation. The payoff matrix is the following:

$$\begin{array}{cc} & \begin{array}{cc} S & H \end{array} \\ \begin{array}{c} S \\ H \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

where S and H stand respectively for Stag and Hare and $a > c \geq d > b$. Coordination games are characterized by two strict, pure strategy NEs and one non-strict, mixed strategy NE, which are, respectively, the risk dominant

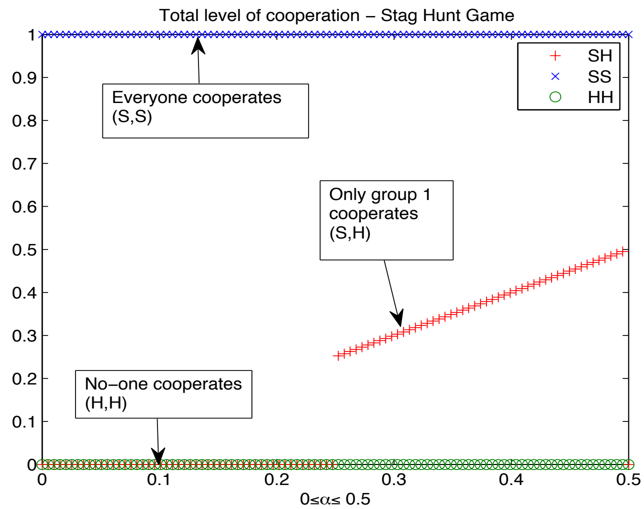

Figure 2.2: The global level of cooperation in the two-groups population for the different GESSs, as a function of α.

equilibrium (H, H), the payoff dominant one (S, S) and the symmetric mixed equilibrium with $q_1^* = q_2^* = \frac{d-b}{a-b-c+d}$.

We set here $a = 2$, $b = 0$, $c = 1$, $d = 1$ and we determine the equilibria of the two groups game as a function of α. We find that the strict GESSs and the strict NEs don't coincide, since we obtain strict GESSs but not NEs. The two-groups Stag-Hunt Game only have the pure-pure strict NE (S, S), for all values of α, while for the GESSs we obtain that:

- for $0 < \alpha < 0.5$ the game admits two pure-pure strong GESSs: (S, S) and (H, H);
- for $0.25 < \alpha < 0.5$ the game admits three pure-pure strong GESSs: (S, S) and (H, H) and (S, H);
- the game doesn't admit any strict mixed NE.

2.4.3 Prisoner's Dilemma

We consider another classical example in game theory, the Prisoner's Dilemma, which belongs to a third class of games, the *pure dominance class*.

The story to imagine is the following: two criminals are arrested and separately interrogated; they can either accuse the other criminal, either remain silent. If both of them accuse the other (defect - D), then they will be both imprisoned for 2 years. If only one accuse the other, the accused is punished with 3 years of jail while the other is free. If both remain silent (cooperate -C), each of them will serve only one year in jail. The corresponding payoff matrix is the following:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

where $c > a > d > b$.

In standard GT, pure dominance class games have a unique pure, strict and symmetric NE, which also is the unique ESS; in the Prisoner's Dilemma it's (D, D).

We set $a = 2$, $b = 0$, $c = 3$, $d = 1$ and we study the two-groups corresponding game. As in the previous example, we find strict GESSs which are not strict NEs. In particular, we have that:

- (C, C) is always a GESS and a strict NE for all values of α;
- (D, D) is a GESS for all values of α but it is never a strict NE;
- (C, D) (symmetrically (D, C)) is always a GESS and a strict NE for $0.5 < \alpha < 1$ (symmetrically $0 < \alpha < 0.5$);
- the game doesn't admit mixed GESSs;

For any value of α, the two groups game thus admits three pure GESSs and two pure strict NEs .

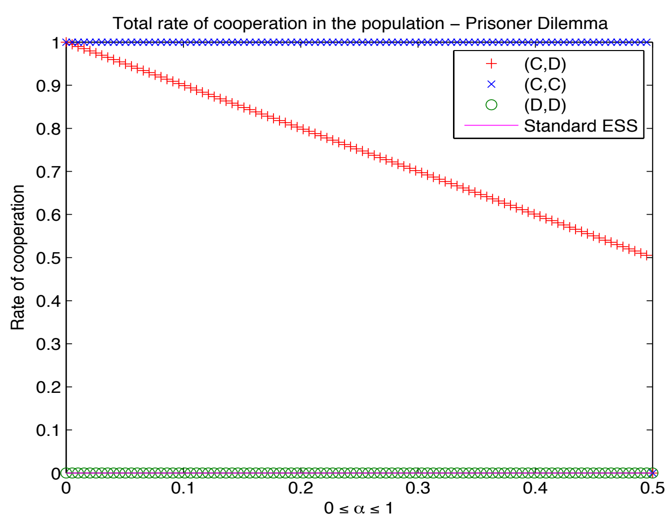

Figure 2.3: Total rate of collaboration for different GESSs as a function of group 1 size α in the prisoner's Dilemma

2.5 Multiple Access Control

In this section we shortly introduce a possible refinement of our model, which will be further investigated in future works. The idea is to distinguish between interactions among members of the same group and of two different ones. We study a particular example where we modify the group utility function defined in (2.5), supposing that the immediate payoff matrix depends on the groups the two players belong to (if its the same one or they belong to two different groups).

The utility function of a group i playing q_i against a population profile q_{-i} can be written as follows:

$$U(\mathbf{q_i}, \mathbf{q_{-i}}) = \alpha_i K(\mathbf{q_i}, \mathbf{q_i}) + \sum_{j \neq i} \alpha_i J(\mathbf{q_i}, \mathbf{q_j}), \tag{2.15}$$

where by $K(\mathbf{p}, \mathbf{q})$ we denote the immediate expected payoff of an individual playing p against a member of its own group using q, and by $J(\mathbf{p}, \mathbf{q})$ the immediate expected payoff associated to interactions among individuals of different groups.

We consider a particular application of this model in Aloha system, such that a large population of mobile phones interfere with each other through local pairwise interactions. The population of mobiles is decomposed into N groups G_i, $i = 1, 2, .., N$ of normalized size α_i with $\sum_{j=1}^{N} \alpha_i = 1$ and each mobile can decide either to transmit (T) or to not transmit (S) a packet to the receiver within transmission range. The interferences occur according to the Aloha protocol, which assume that if more than one neighbor of a receiver transmits a packet at the same time this causes a collision and the failure of transmission. The channel is assumed to be ideal for transmission and the only errors occurring are due to these collisions.
We denote by μ the probability that a mobile k has its receiver $R(k)$ in its range. When the mobile k transmits to the receiver $R(k)$, all the other mobiles within a circle of radius R centered on $R(k)$ cause interference to k and thus the failure of mobile's k packet transmission to $R(k)$.

A mobile belonging to a given group i may use a mixed strategy $\mathbf{p}_i = (p_i, 1 - p_i)$, where p_i (resp. $1 - p_i$) is the probability to choose the action (T) (resp (S)). Let γ be the probability that a mobile is alone in a given local interaction; before transmission, the tagged mobile doesn't know if if there is another transmitting mobile within its range of transmission.

Let P_1 (resp. P_2) be the immediate payoff matrix associated to interactions among mobiles belonging to the same group (resp. to two different ones):

$$P_1 \equiv \begin{array}{c c} & \begin{array}{cc} T & S \end{array} \\ \begin{array}{c} T \\ S \end{array} & \begin{pmatrix} -2\delta & 1-\delta \\ 1-\delta & 0 \end{pmatrix} \end{array}, \qquad P_2 \equiv \begin{array}{c c} & \begin{array}{cc} T & S \end{array} \\ \begin{array}{c} T \\ S \end{array} & \begin{pmatrix} \delta & 1-\delta \\ 0 & 0 \end{pmatrix} \end{array}.$$

where $0 < \delta < 1$ is the cost of transmitting the package. The definition of the matrix P_1 implies that, when two mobiles of the same group i interfere, any successful transmission is equally rewarding for the group i. The resulting expected payoffs of a mobile playing $\mathbf{q}_i$ against a member belonging to its own group and to a different one, using respectively the same strategy $\mathbf{q}_i$ and a different one $\mathbf{q}_j$ are, respectively, the following:

$$\begin{aligned}K(\mathbf{q}_i,\mathbf{q}_i) &= \mu\,[q_i[\gamma(1-\delta)+(1-\gamma)((1-\delta)(1-q_i)-2\delta q_i)]\\ &\quad +(1-\gamma)(1-\delta)(1-q_i)q_i]\\ &= \mu q_i[(1-\delta)(2-\gamma)-2(1-\gamma)q_i]\end{aligned}$$

$$\begin{aligned}J(\mathbf{q}_i,\mathbf{q}_j) &= \mu q_i[\gamma(1-\delta)+(1-\gamma)((1-\delta)(1-q_j)-\delta q_j]\\ &= \mu q_i[1-\delta-(1-\gamma)q_j]\end{aligned}$$

The expected utility of group i is then given by:

$$U(\mathbf{q}_i,\mathbf{q}_{-i}) = \mu q_i[1-\delta+(1-\gamma)(\alpha_i(1-\delta-q_i)-\sum_{j=1}^{N}\alpha_j q_j)] \tag{2.16}$$

By following the same line of analysis followed in section 2.2, the strategy $\mathbf{q}$ is a GESS if $\forall i = 1,\dots N$ the two following conditions are satisfied:

1. $F_i'(\mathbf{p}_i,\mathbf{q}) \equiv (q_i-p_i)[1-\delta+(1-\gamma)(\alpha_i(1-\delta-2q_i)-\sum_{j=1}^{N}\alpha_j q_j)] \geq 0 \quad \forall p_i,$
2. If $F_i'(\mathbf{p}_i,\mathbf{q}) = 0$ for some $p_i \neq q_i$, then $(p_i-q_i)^2(1-\gamma)\alpha_i > 0 \quad \forall p_i \neq q_i$.

We observe that, since the inequality $(p_i-q_i)^2(1-\gamma)\alpha_i > 0$ holds for all values of the parameters, the second condition is always satisfied and thus the first condition is sufficient for the existence of a GESS. We now characterize the GESSs of the presented MAC game. With no loss of generality, we reorder the groups so that $\alpha_1 \leq \alpha_2 \ldots \leq \alpha_N$.

Proposition 5. *In the presented MAC game, we find that:*

- *The pure symmetric strategy $(S,\dots,S)$ is never a GESS.*
- *If a group G_i adopts pure strategy S, then at the equilibrium, all smaller groups also use S.*
- *If a group G_i adopts pure strategy T, then at the equilibrium, all smaller groups transmit. If the bigger group G_N use strategy T at the equilibrium, then $\gamma > \bar{\gamma}$.*
- *If a group G_i adopts an equilibrium mixed strategy $q_i \in]0,1[$, then if $q_i > \frac{1-\delta}{2}$, at the equilibrium all smaller groups use pure strategy T, whereas if $q_i < \frac{1-\delta}{2}$, smaller groups play S.*
- *The unique fully mixed GESS of the game is $\mathbf{q}^* = (q_1^*,\dots,q_N^*)$, given by:*

$$q_i^* = \frac{(1-\delta)(1+\gamma+(1-\gamma)(2+N)\alpha_i)}{2(N+2)(1-\gamma)\alpha_i} \tag{2.17}$$

under the condition: $\gamma < \underline{\gamma}$.

The thresholds $\underline{\gamma}$ and $\bar{\gamma}$ are defined as follows:

$$\underline{\gamma} \equiv \min_{\alpha_i} \frac{\alpha_i(N+2)(1+\delta)-(1-\delta)}{\alpha_i(N+2)(1+\delta)+(1+\delta)},$$

$$\bar{\gamma} \equiv \max_{\alpha_i} \left(1 - \frac{1-\delta}{\alpha_i(\delta+1)+1}\right).$$

Proof. The proof is available in [60]. □

In order to provide a better insight, we consider a two groups MAC game, in which we set a low value of the cost of transmission, $\delta = 0.2$, and groups' sizes $\alpha_1 = \alpha = 0.4$, $\alpha_2 = 1 - \alpha = 0.6$, and we let vary the value of the parameter γ. We obtain three different equilibria, depending on the value of γ: a pure GESS $\mathbf{q}_P^*$, a fully mixed GESS $\mathbf{q}_M^*$ and a pure-mixed one $\mathbf{q}_{PM}^*$. In figure 2.4 we plot the fully mixed and the pure GESS. For $\gamma < \underline{\gamma} = 0.3$ the game admits a GESS $\mathbf{q}^*\mathbf{q}_M^* = (q_1^*, q_2^*)$, whose components are shown in the plot. Then, for $\gamma = \bar{\gamma} > 0.53$, $\mathbf{q}^*\mathbf{q}_P^* = (T, T)$. We also represent the value of $q_{std}^* := \min(1, \frac{1}{1-\gamma} - \Delta$. We note that fully mixed equilibrium strategies adopted by the two groups, q_1^*, q_2^*, are both lower then q_{std}^*.

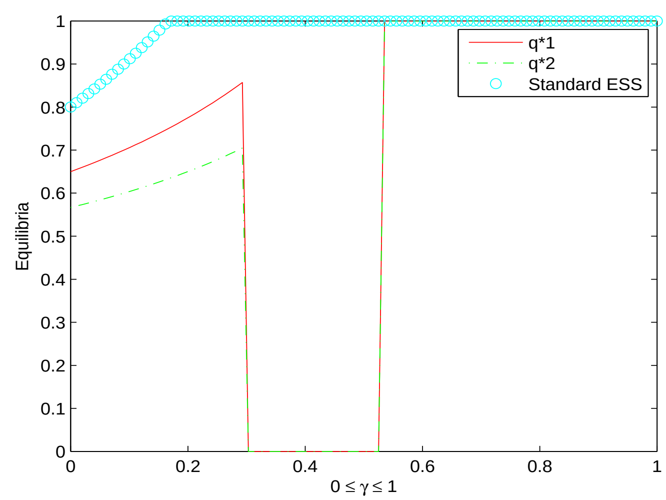

Figure 2.4: The value of the equilibrium strategy q_1^* and q_2^* in a two groups MAC game as a function of γ for $\alpha = 0.4$ compared to q_{std}^* .

In figure 2.5 we plot the value of the second mixed component of the pure-mixed GESS of the game: (T, q_T), which is an equilibrium in the interval $0 \leq \gamma < 0.4$, and we compare it to q_{std}^*. We observe that, for the second group the probability of transmitting is always lower w.r.t. the standard game.

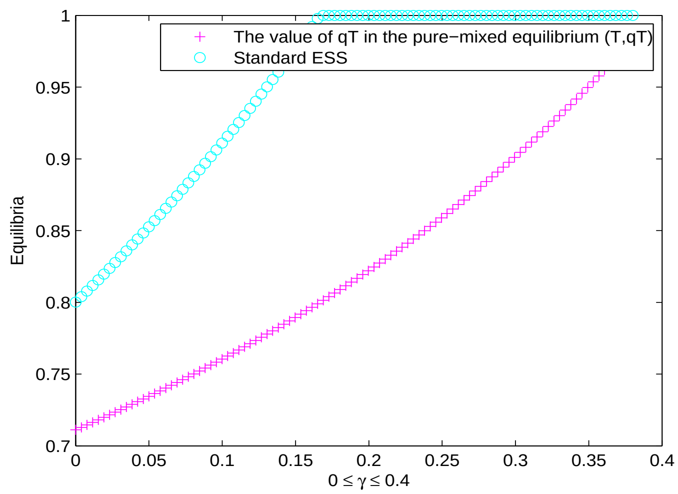

Figure 2.5: The value of the equilibrium strategy q_2^* of the second group in the pure-mixed equilibrium (T, qT) as a function of γ for $\alpha = 0.85$ compared to q_{std}^* .

We denote by $p_S(\mathbf{p})$ the probability of a successful transmission in a population whose profile is $\mathbf{p}$. If $N = 2$, we obtain that:

$$
\begin{aligned}
p_S(\mathbf{p}) = \mu[\gamma(\alpha p_1 + (1-\alpha)p_2)] + (1-\gamma)(2\alpha^2 p_1(1-p_1) + \\
+ \alpha(1-\alpha)((1-p_2)p_1 + (1-p_1)p_2) + 2(1-\alpha)^2 p_2(1-p_2))].
\end{aligned}
$$

In figure 2.6 we represent the value of the equilibrium $p_S^* = p_S(\mathbf{q}_M^*)$ as a function of γ for $\alpha = 0.4$. We observe that, $\gamma < \underline{\gamma}$, even if at the equilibrium the probability of the transmission is lower in the groups game .

2.6 Conclusions

In this work we defined the new concept of the GESS, an Evolutionarily Stable Strategy in a group-players context and we studied its relationship and the Nash equilibrium and with the standard ESS. We studied some classical examples of two players and two strategies games transposed in our group-players framework and we found that the fact of considering interactions among individuals maximizing their group's utility (instead of their own one) impacts individuals' behavior and changes the structure of the equilibria. We briefly introduce a refinement of our model, where we redefine the utility of a group in order to consider different utilities for interactions among members of the same group or of a different one. Through a particular example in MAC games we showed how the presence of groups can encourage cooperative behaviors. There are many possible issues to be developed in future studies. We are

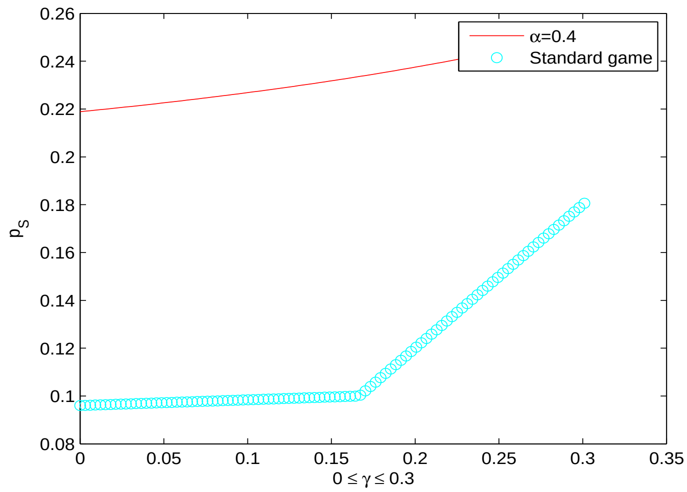

Figure 2.6: The probability $p_S(\mathbf{q}^*_M)$ of a successful transmission for the fully mixed GESS in a two groups MAC game at the equilibrium as a function of γ .

currently studying the replicator dynamics in this group-players games, in order to investigate the existing relationship between the rest point of such dynamics and our GESS, following the lines off he Folk Theorem of EG. At a more theoretical level, the concept of group's utility could be further investigated, and the coexistence of both selfish and altruistic behaviors among individuals could be taken into account.